S množinami okolo sveta

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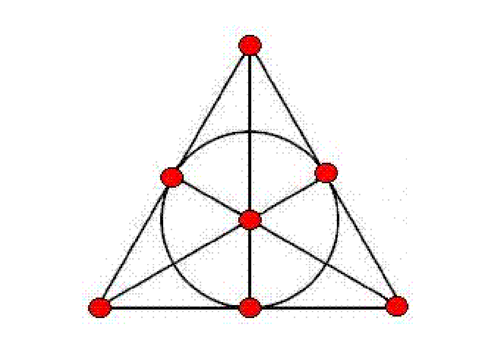
Klasický príklad na grupách

Príklad:
$G = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$.
Množina $D = \{1, 2, 4\} \subseteq \mathbb{Z}_7$

$$2 - 1 = \mathbf{1} \qquad 4 - 2 = \mathbf{2} \qquad 4 - 1 = \mathbf{3}$$
$$1 - 4 = \mathbf{4} \qquad 2 - 4 = \mathbf{5} \qquad 1 - 2 = \mathbf{6}$$

D je $(7, 3, 1)$ diferenčná množina.
Je to aj $(7, 3, 1)$ dizajn.

Fanova Rovina



Klasický príklad na grupách

Príklad:
$G = \mathbb{Z}_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
Množina $S = \{1, 3, 9, 5, 4\} \subseteq \mathbb{Z}_{11}$

Multimnožina:

$$D(S) = \{9, 3, 7, 8, 2, 5, 9, 10, 8, 6, 4, 5, 4, 2, 7, 1, 3, 1, 6, 10\} =$$

$$= \{1,1,2,2,3,3,4,4,5,5,6,6,7,7,8,8,9,9,10,10\}$$

Teda S je $(11, 5, 2)$ diferenčná množina.

Klasický príklad na grupách

- klasický kombinatorický koncept
- *symetrické dizajny*
- diferenčné množiny sa používajú pri kódovaní na konštrukciu vector codebook

Construction

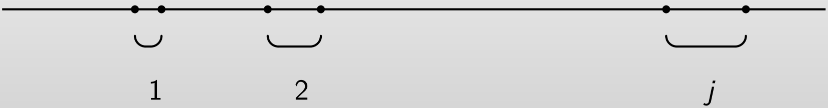
Let λ be a positive integer.
Take $\{M_n\}_{n=1}^{\infty}$ to be the sequence of subsets of $\mathbb{N}$ defined recursively as follows :

1. $M_1 = \{\ m_0,\ m_0\ +\ 1\ \}$, *where m_0 is an arbitrary element of $\mathbb{N}$;*
2. $$M_{n+1} = M_n \cup \{2(k+1), 2(k+1)+j\}$$
   where $k = \max M_n$ and j is the smallest positive integer which appears in $D(M_n)$ fewer than λ times

Then, the set

$$M_\lambda = \bigcup \{M_n \mid n \in \mathbb{N}\}$$

is a λ-difference set on $\mathbb{N}$.



- všetky *nové* rozdiely sú väčšie ako staré
- nové rozdiely sa objavia *nanajvýš* 2 krát
- Táto konštrukcia funguje vďaka odláčaniu problémov ďaleko

Generalized Difference Sets on Positive Integers

Definition

Let Λ be a sequence of nonnegative integers

$$\{\lambda_i\}_{i=1}^{\infty}.$$

A *Λ-generalized difference set* S is a subset of $\mathbb{N}$ such that each positive integer i appears as a difference $s - s'$ of elements from S exactly λ_i times.

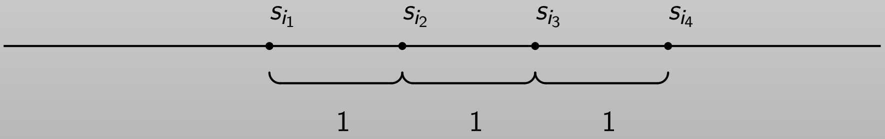
Sequence Λ is called the *frequency sequence of* S.

Non-Existing Example

Uvažujme postupnosť:

$$\Lambda = \{3, 1, 1, 1, \ldots\}$$

Prípad 1.



Non-Existing Example

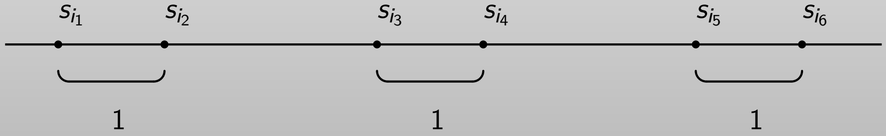
$$\Lambda = \{3, 1, 1, 1, \ldots\}$$

Prípad 2.

Non-Existing Example

$$\Lambda = \{3, 1, 1, 1, \ldots\}$$

Prípad 3.



Existenčná Veta

Theorem

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ *be a sequence of positive integers such that* $\lambda_i \geq 2$ *for all but finitely many* $i \in \mathbb{N}$*. Then there exists a generalized difference set* S *of type* Λ*.*

Vynechané:

Λ's that contain infinitely many entries equal to 1

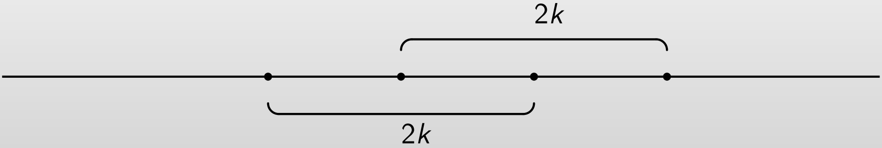
- the sequences Λ with infinitely many $\lambda_i = 1$ but only finitely many $\lambda_i \geq 2$
- the sequences Λ with infinitely many $\lambda_i = 1$ as well as infinitely many $\lambda_i \geq 2$.

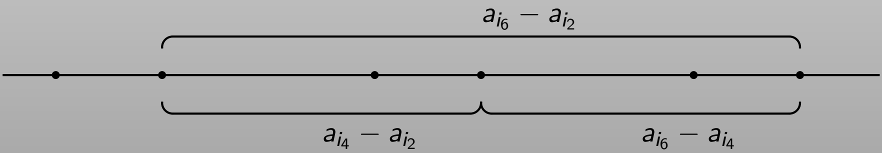
Theorem (TJ, R.Jajcay)

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ *be a frequency sequence consisting entirely of 1's and 2's, containing infinitely many 2's. Then there exists a generalized difference set* D *of type* Λ*.*

Príklad:

$$\Delta = \{3, 1, 3, 1, 3, 1, \ldots\}$$





Example

$$\Delta = \{1, 3, 1, 3, 1, 3, \ldots\}$$

Theorem (Swara Kopparty)

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a frequency sequence consisting of 0's and 1's containing only finitely many 0's. Then there exists a generalized difference set S of type Λ.

Theorem (Martin Štefaňák)

“ Triangular algorithm"for finite frequency sequences.

Density questions

s_n grows very fast with respect to n

In all the sequences we have constructed so far

$$s_n \geq 2^{n/2}$$

So all the sequences constructed so far have an exponential growth. On the other hand, the obvious lower bound gives

$$\frac{n(n-1)}{2} \leq \sum_{i=1}^{s_n-1} \lambda_i.$$

Example

$$\Lambda = \{1, 1, 1, 1, \ldots\}$$

$$n(n-1)/2 < s_n$$

Any generalized difference set of type $\{1, 1, 1, \ldots\}$ has to grow at the magnitude $\Omega(n^2)$

Greedy Construction

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a sequence of positive integers, and let $\{M_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{N}$ defined recursively as follows :

1. $M_1 = \{m_1, m_1 + 1\}$, where m_1 is an arbitrary positive integer;
2. let j be the smallest positive integer that appears in $Df(M_n)$ less than λ_j times, the set M_{n+1} is defined from the set M_n by setting

   $$M_{n+1} = M_n \cup \{m_{n+1}, m_{n+1} + j\},$$

   where m_{n+1} is the *smallest* positive integer not belonging to M_n and satisfying the property $\Lambda(M_n \cup \{m_{n+1}, m_{n+1} + j\}) \leq \Lambda$.

Greedy Construction

A computer run for S_Λ, with $\Lambda = \{1, 1, 1, 1, \ldots\}$, appears to indicate the rate of growth of about n^4.

Supergreedy Construction

Let $\Lambda = \{\lambda_i\}_{i=1}^{\infty}$ be a sequence of positive integers, and let $\{S_n\}_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{N}$ defined recursively as follows :

1. $S_1 = \{s_1\}$, where s_1 is an arbitrary positive integer;
2. the set S_{n+1} is defined from the set S_n by setting

   $$S_{n+1} = S_n \cup \{s_{n+1}\},$$

   where s_{n+1} is the *smallest* positive integer not belonging to S_n that satisfies the property $\Lambda(S_n \cup \{s_{n+1}\}) \leq \Lambda$.

Then $S_\Lambda = \bigcup\{S_n \,|n \in \mathbb{N}\}$.

Supergreedy Construction

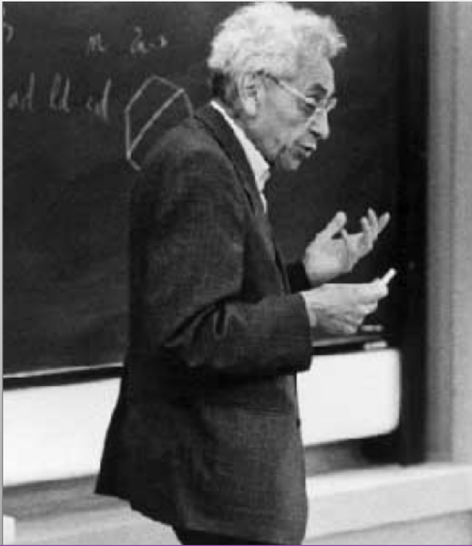
- The differences are not constructed successively in an increasing order
- The rate of growth is the “slowest possible", at least at the beginning
- There is no guarantee, that the differences that are being skipped will eventually be filled

Supergreedy Construction

- In a computer simulation, we computed the first elements of S_Λ , $\Lambda = \{1, 1, 1, 1, \ldots\}$.
  In a computation up to $s_i \geq 10,000,000$, the smallest difference that was still missing was 33.
- **It is however impossible to draw any conclusions** – there were several other small numbers missing, but when we got all the way up to $s_i \geq 1,000,000$, they were filled.
- We do not know at this point, *whether the supergreedy algorithm constructs the desired generalized difference set*.
- pekná práca Ivany Kellyérovej

Príbuzný problém Paula Erdős

Počas Erdős ovej návštevy v Lincolne, NE, Paul Erdős zadal nasledujúci problém:



Let S be a set of integers, and let r_i denote the number of different ways in which the positive integer i appears as a sum of two (not necessarily distinct) elements from S.

Is there a set S with the property $r_i \geq 1$, for all $i \in \mathbb{N}$, such that $\overline{\lim}_{i \to \infty} r_i < \infty$?

Theorem (TJ, R. Jajcay)

There exists a set of integers S such that each positive number appears as a sum of two elements from S exactly once.

Construction

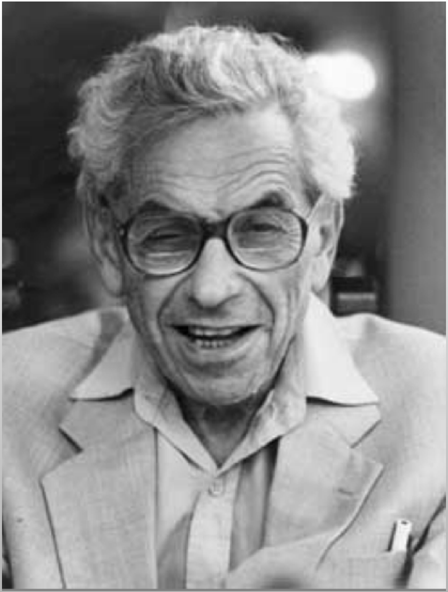
1. $A_1 = \{0, 1\}$;
2. *let* j *be the smallest positive integer that does not appear as a sum of two elements from* A_n*, then the set* A_{n+1} *is defined from the set* A_n *by setting*

$$A_{n+1} = A_n \cup \{a_{n+1} + j, -a_{n+1}\},$$

where $a_{n+1} = 4 \cdot max\{|a_i| \mid a_i \in A_n\} + 1$.

Denote $S = \bigcup\{A_n \,|n \in \mathcal{N}\}$.

“But I meant to say that S is a set of positive integers"



Problem of Erdős - Correct Version

Let S be a set of positive integers, and let r_i denote the number of different ways in which the positive integer i appears as a sum of two (not necessarily distinct) elements from S.

Is there a set S with the property $r_i \geq 1$, for all $i \in \mathbb{N}$, such that $\overline{\lim}_{i \to \infty} r_i < \infty$?